

# A DERIVED CATEGORY APPROACH TO KEMPF'S VANISHING THEOREM

ALEXANDER SAMOKHIN

ABSTRACT. We give a proof of the "Frobenius-eigenvalue" property of the Steinberg line bundle that implies Kempf's vanishing theorem. Our argument is based on the structure of derived categories of coherent sheaves on flag varieties.

## 1. Introduction

Let  $\mathbf{G}$  be a split semisimple simply connected algebraic group over a perfect field  $k$  of characteristic  $p$ . The weight  $-\rho = -\sum \omega_i$ , where  $\omega_i$  are the fundamental weights of  $\mathbf{G}$ , is known to play a fundamental rôle in representation theory of  $\mathbf{G}$ . For  $q = p^n, n \geq 1$ , the Steinberg weight  $(q-1)\rho$  is equally important in representation theory of semisimple groups in defining characteristic. In particular, there is a remarkable property that the corresponding line bundle  $\mathcal{L}_{(q-1)\rho}$  on the flag variety  $\mathbf{G}/\mathbf{B}$  enjoys: its pushforward under the  $n$ -th iteration of Frobenius morphism is a trivial vector bundle whose space of global sections is canonically identified with the Steinberg representation  $\mathrm{St}_q$ :

$$(1) \quad F_*^n \mathcal{L}_{(q-1)\rho} = \mathrm{St}_q \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}}.$$

This was proven independently and at around the same time by Andersen in [1] and by Haboush in [5]. Back to the weight  $-\rho$ , isomorphism of vector bundles (1) is equivalent to saying that the line bundle  $\mathcal{L}_{-\rho}$  is an "eigenvector" with respect to Frobenius morphism, i.e.  $F_*^n \mathcal{L}_{-\rho} = \mathrm{St}_q \otimes \mathcal{L}_{-\rho}$ . This fact has many important consequences for representation theory of algebraic groups in characteristic  $p$ : in particular, the Kempf vanishing theorem [8] easily follows from it (see [1] and [5]). The proofs of *loc.cit.* were essentially representation-theoretic. The goal of this note is to prove isomorphism (1) using the structure of the derived category of coherent sheaves on the flag variety  $\mathbf{G}/\mathbf{B}$ . The idea in a nutshell is as follows. Given a smooth algebraic variety  $X$  and a semiorthogonal decomposition  $\langle \mathbf{D}_0, \mathbf{D}_1 \rangle$  of the derived category  $\mathrm{D}^b(X)$  (see Section 2 for the details), any object of  $\mathrm{D}^b(X)$  – in particular, any vector bundle  $\mathcal{F}$  on  $X$ , can be decomposed with respect to  $\mathbf{D}_0$  and  $\mathbf{D}_1$ . Thus, if  $\mathcal{F}$  is right orthogonal to  $\mathbf{D}_1$ , i.e.  $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathbf{D}_1, \mathcal{F}) = 0$ , it automatically belongs to  $\mathbf{D}_0$ . It turns out that for a semiorthogonal decomposition of the derived category  $\mathrm{D}^b(\mathbf{G}/\mathbf{B})$  into two pieces, one of which is the admissible subcategory  $\langle \mathcal{L}_{-\rho} \rangle$  generated by the single line bundle  $\mathcal{L}_{-\rho}$  and the other one being its left orthogonal  ${}^\perp \langle \mathcal{L}_{-\rho} \rangle$ , the bundle  $F_*^n \mathcal{L}_{-\rho}$  is right orthogonal to  ${}^\perp \langle \mathcal{L}_{-\rho} \rangle$ . Therefore, it should belong to the subcategory  $\langle \mathcal{L}_{-\rho} \rangle$ . Being generated by a single exceptional bundle, the latter subcategory is equivalent to the derived category of vector spaces over  $k$ ; thus, one has  $F_*^n \mathcal{L}_{-\rho} = \mathcal{L}_{-\rho} \otimes \mathbf{V}$  for some graded vector space  $\mathbf{V}$ . Since the left hand side of this isomorphism is a vector bundle, i.e. a pure object of  $\mathrm{D}^b(\mathbf{G}/\mathbf{B})$ , the graded vector

space  $V$  should only have a non-trivial zero degree part, which is a vector space of dimension  $q^{\dim(\mathbf{G}/\mathbf{B})}$ . Tensoring the both sides with  $\mathcal{L}_{-\rho}$  and taking the cohomology, one obtains an isomorphism  $V = H^0(\mathbf{G}/\mathbf{B}, \mathcal{L}_{(q-1)\rho}) = \mathrm{St}_q$ , and hence isomorphism (1).

Unfolding this argument takes the rest of the note. The key step, which may have an independent interest, consists of proving a special property of the semiorthogonal decomposition described above that allows to easily check the orthogonality properties for the bundle  $F_*^n \mathcal{L}_{-\rho}$ . This is done in Lemma 3.1. Theorem 4.1, which is equivalent to isomorphism (1), immediately follows from it.

However, so far there is a drawback in the above argument, if one is bound to establish an independent proof of the Kempf vanishing. Specifically, claiming that the line bundle  $\mathcal{L}_{-\rho}$  on the flag variety is exceptional, which eventually allows to produce the desired semiorthogonal decomposition and to establish isomorphism (1), is equivalent to higher cohomology vanishing of the structure sheaf of flag variety. One way to prove this is via the Kodaira vanishing theorem [3], which obviously holds in the present situation; however, the Kempf vanishing follows immediately from the former vanishing theorem (see Section 2.3 and Proposition 3.1). Alternatively, one can use the fact that Schubert varieties have rational singularities in characteristic  $p$  (see [11]). This implicitly contains the Frobenius splitting property of flag varieties [9] that as well implies the Kempf vanishing. Since in the last step in the proof of Lemma 3.1 we have to also resort to rationality of singularities of Schubert varieties, we keep to the vanishing theorem for these [11] to reduce using powerful vanishing statements to a bare minimum. Otherwise, the only cohomology vanishing that is used in our argument is the most elementary part of Bott's vanishing [4] that holds over  $\mathbb{Z}$  (see Section 2.2 for the details).

The present note was motivated by the author's computations of Frobenius pushforwards of homogeneous vector bundles on flag varieties. The derived localization theorem of [2] implies, in particular, that for a regular weight  $\chi$  (that is, for a weight having trivial stabilizer with respect to the dot-action of the (affine) Weyl group) the bundle  $F_* \mathcal{L}_\chi$  is a generator in the derived category  $D^b(\mathbf{G}/\mathbf{B})$ ; in other words, there are sufficiently many indecomposable summands of  $F_* \mathcal{L}_\chi$  to generate the whole derived category  $D^b(\mathbf{G}/\mathbf{B})$ . Knowing indecomposable summands of these bundles (e.g., for  $p$ -restricted weights) may clarify, in particular, cohomology vanishing patterns of line bundles on  $\mathbf{G}/\mathbf{B}$ . On the contrary, the weight  $-\rho$  being the most singular, the thick category generated by the bundle  $F_* \mathcal{L}_{-\rho}$  "collapses" to the subcategory generated by the single line bundle  $\mathcal{L}_{-\rho}$ , which is encoded in isomorphism (1).

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**Notation.** Given a split semisimple simply connected algebraic group  $\mathbf{G}$  over a perfect field  $k$ , let  $\mathbf{T}$  denote a maximal torus of  $\mathbf{G}$ , and let  $\mathbf{T} \subset \mathbf{B}$  be a Borel subgroup containing  $\mathbf{T}$ . The flag variety of Borel subgroups in  $\mathbf{G}$  is denoted  $\mathbf{G}/\mathbf{B}$ . Denote  $X(\mathbf{T})$  the weight lattice, and let  $R$  and  $R^\vee$  denote the root and coroot lattices, respectively. Let  $S$  be the set of simple roots relative to the choice of a Borel subgroup than contains  $\mathbf{T}$ . For a simple root  $\alpha \in S$ , denote  $\mathbf{P}_\alpha$  the minimal parabolic subgroup of  $\mathbf{G}$  associated to  $\alpha$ . Given a weight  $\lambda \in X(\mathbf{T})$ , denote  $\mathcal{L}_\lambda$  the corresponding line bundle on  $\mathbf{G}/\mathbf{B}$ . All the functors are supposed to be derived, i.e., given a morphism  $f : X \rightarrow Y$  between two schemes, we write  $f_*, f^*$  for the corresponding derived functors of push-forwards and pull-backs.

## 2. Some preliminaries

**2.1. Flag varieties of Chevalley groups over  $\mathbb{Z}$ .** Let  $\mathbb{G} \rightarrow \mathbb{Z}$  be a semisimple Chevalley group scheme (a smooth affine group scheme over  $\text{Spec}(\mathbb{Z})$  whose geometric fibres are connected semisimple algebraic groups), and  $\mathbb{G}/\mathbb{B} \rightarrow \mathbb{Z}$  be the corresponding Chevalley flag scheme (resp.,  $\mathbb{P} \subset \mathbb{G}$  the corresponding parabolic subgroup scheme over  $\mathbb{Z}$ ). Then  $\mathbb{G}/\mathbb{P} \rightarrow \text{Spec}(\mathbb{Z})$  is flat and the line bundle  $\mathcal{L}$  on  $\mathbf{G}/\mathbf{P}$  also comes from a line bundle  $\mathbb{L}$  on  $\mathbb{G}/\mathbb{P}$ . Let  $k$  be a field of arbitrary characteristic, and  $\mathbf{G}/\mathbf{B} \rightarrow \text{Spec}(k)$  be the flag variety obtained by base change along  $\text{Spec}(k) \rightarrow \text{Spec}(\mathbb{Z})$ .

**2.2. Cohomology of line bundles on flag varieties.** We recall first the classical Bott's theorem (see [4]). Let  $\mathbb{G} \rightarrow \mathbb{Z}$  be a semisimple Chevalley group scheme as above. Assume given a weight  $\chi \in X(\mathbf{T})$ , and let  $\mathcal{L}_\chi$  be the corresponding line bundle on  $\mathbb{G}/\mathbb{B}$ . The weight  $\chi$  is called *singular*, if it lies on a wall of some Weyl chamber defined by  $\langle -, \alpha^\vee \rangle = 0$  for some coroot  $\alpha^\vee \in R^\vee$ . Weights, which are not singular, are called *regular*. Let  $k$  be a field of characteristic zero, and  $\mathbf{G}/\mathbf{B} \rightarrow \text{Spec}(k)$  the corresponding flag variety over  $k$ . The weight  $\chi \in X(\mathbf{T})$  defines a line bundle  $\mathcal{L}_\chi$  on  $\mathbf{G}/\mathbf{B}$ .

**Theorem 2.1.** [4, Theorem 2]

- (a) If  $\chi + \rho$  is singular, then  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = 0$  for all  $i$ .
- (b) If  $\chi + \rho$  is regular and dominant, then  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = 0$  for  $i > 0$ .
- (c) If  $\chi + \rho$  is regular, then  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) \neq 0$  for the unique degree  $i$ , which is equal to  $l(w)$ . Here  $l(w)$  is the length of an element of the Weyl group that takes  $\chi$  to the dominant chamber, i.e.  $w \cdot \chi \in X_+(\mathbf{T})$ . The cohomology group  $H^{l(w)}(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi)$  is the irreducible  $\mathbf{G}$ -module of highest weight  $w \cdot \chi$ .

**Remark 2.1.** Some bits of Theorem 2.1 are still true over  $\mathbb{Z}$ : if a weight  $\chi$  is such that  $\langle \chi + \rho, \alpha^\vee \rangle = 0$  for some simple root  $\alpha$ , then the corresponding line bundle is acyclic. Indeed, Lemma from [4, Section 2] holds over fields of arbitrary characteristic. Besides this, however, very little of Theorem 2.1 holds over  $\mathbb{Z}$  (see [7, Part II, Chapter 5]).

From now on, unless specified otherwise, the base field  $k$  is assumed to be a perfect field of characteristic  $p > 0$ .

**Proposition 2.1.** If a line bundle  $\mathcal{L}_\chi$  is acyclic on  $\mathbf{G}/\mathbf{B}$ , i.e.  $H^*(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = 0$ , then the weight  $\chi + \rho$  is singular.

*Proof.* Assume the contrary, that is the weight  $\chi + \rho$  is regular. By Theorem 2.1, (c), the line bundle  $\mathcal{L}_\chi$  in characteristic 0 has a non-vanishing cohomology group in some degree. By semicontinuity, this would mean that in characteristic  $p$  there would be a non-trivial cohomology group in the same degree, which is a contradiction.  $\square$

**2.3. Kempf's vanishing theorem.** Kempf's vanishing theorem, originally proven by Kempf in [8], and subsequently by Andersen [1] and Haboush [5] with shorter representation-theoretic proofs (see also [7, Part II, Chapter 4]), states that given a dominant weight  $\chi \in X(\mathbf{T})$ , i.e. a weight, such that  $\langle \chi, \alpha^\vee \rangle \geq 0$  for all simple coroots  $\alpha^\vee$ , the cohomology groups  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi)$  vanish in positive degrees, i.e.  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = 0$  for  $i > 0$ . This theorem is ubiquitous in representation theory of algebraic groups in characteristic  $p$ . For convenience of the reader, we briefly recall how it can be obtained from the main isomorphism  $F_*^n \mathcal{L}_{(q-1)\rho} = \mathrm{St}_q \otimes \mathcal{O}_{\mathbf{G}/\mathbf{B}}$  (recall that  $q = p^n$  for  $n \in \mathbb{N}$ ). From  $\langle \chi, \alpha^\vee \rangle \geq 0$  one obtains  $\langle \chi + \rho, \alpha^\vee \rangle > 0$  for all simple coroots  $\alpha^\vee$ . By [7, Part II, Proposition 4.4], the line bundle  $\mathcal{L}_{\chi+\rho}$  is ample on  $\mathbf{G}/\mathbf{B}$ . Consider the weight  $q(\chi + \rho) - \rho = q\chi + (q-1)\rho$ . Since  $\mathcal{L}_{\chi+\rho}$  is ample, one can choose  $n \in \mathbb{N}$  large enough so that the line bundle  $\mathcal{L}_{q(\chi+\rho)}$  be very ample. From the well-known properties of the Frobenius morphism it then follows

$$(2) \quad H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_{q\chi+(q-1)\rho}) = H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi \otimes F_*^n \mathcal{L}_{(q-1)\rho}) = H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) \otimes \mathrm{St}_q.$$

Now the left hand side group vanishes for  $i > 0$  by Serre's vanishing, the line bundle  $\mathcal{L}_{q(\chi+\rho)}$  being very ample. Hence,  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = 0$  for  $i > 0$  as well.

**2.4. Derived categories of coherent sheaves.** The content of this section can be found, e.g., in [6, Section 1.2].

Let  $k$  be a field. Assume given a  $k$ -linear triangulated category  $\mathbf{D}$ , equipped with a shift functor  $[1]: \mathbf{D} \rightarrow \mathbf{D}$ . For two objects  $A, B \in \mathbf{D}$  let  $\mathrm{Hom}_{\mathbf{D}}^\bullet(A, B)$  be the graded  $k$ -vector space  $\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{D}}(A, B[i])$ . Let  $\mathbf{A} \subset \mathbf{D}$  be a full triangulated subcategory, that is a full subcategory of  $\mathbf{D}$  which is closed under shifts.

**Definition 2.1.** *The right orthogonal  $\mathbf{A}^\perp \subset \mathbf{D}$  is defined to be the full subcategory*

$$(3) \quad \mathbf{A}^\perp = \{B \in \mathbf{D} : \mathrm{Hom}_{\mathbf{D}}(A, B) = 0\}$$

*for all  $A \in \mathbf{A}$ . The left orthogonal  ${}^\perp \mathbf{A}$  is defined similarly.*

**Definition 2.2.** *A full triangulated subcategory  $\mathbf{A}$  of  $\mathbf{D}$  is called right admissible if the inclusion functor  $\mathbf{A} \hookrightarrow \mathbf{D}$  has a right adjoint. Similarly,  $\mathbf{A}$  is called left admissible if the inclusion functor has a left adjoint. Finally,  $\mathbf{A}$  is admissible if it is both right and left admissible.*

If a full triangulated category  $\mathbf{A} \subset \mathbf{D}$  is right admissible then every object  $X \in \mathbf{D}$  fits into a distinguished triangle

$$(4) \quad \dots \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1] \rightarrow \dots$$

with  $Y \in \mathbf{A}$  and  $Z \in \mathbf{A}^\perp$ . One then says that there is a semiorthogonal decomposition of  $\mathbf{D}$  into the subcategories  $(\mathbf{A}^\perp, \mathbf{A})$ . More generally, assume given a sequence of full triangulated subcategories  $\mathbf{A}_1, \dots, \mathbf{A}_n \subset \mathbf{D}$ . Denote  $\langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle$  the triangulated subcategory of  $\mathbf{D}$  generated by  $\mathbf{A}_1, \dots, \mathbf{A}_n$ .

**Definition 2.3.** A sequence  $(\mathbf{A}_1, \dots, \mathbf{A}_n)$  of admissible subcategories of  $\mathbf{D}$  is called *semiorthogonal* if  $\mathbf{A}_i \subset \mathbf{A}_j^\perp$  for  $1 \leq i < j \leq n$ , and  $\mathbf{A}_i \subset {}^\perp \mathbf{A}_j$  for  $1 \leq j < i \leq n$ . The sequence  $(\mathbf{A}_1, \dots, \mathbf{A}_n)$  is called a *semiorthogonal decomposition* of  $\mathbf{D}$  if  $\langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle^\perp = 0$ , that is  $\mathbf{D} = \langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle$ .

**Definition 2.4.** An object  $E \in \mathbf{D}$  of a  $k$ -linear triangulated category  $\mathbf{D}$  is said to be *exceptional* if there is an isomorphism of graded  $k$ -algebras

$$(5) \quad \mathrm{Hom}_{\mathbf{D}}^\bullet(E, E) = k.$$

A collection of exceptional objects  $(E_0, \dots, E_n)$  in  $\mathbf{D}$  is called *exceptional* if for  $1 \leq i < j \leq n$  one has

$$(6) \quad \mathrm{Hom}_{\mathbf{D}}^\bullet(E_j, E_i) = 0.$$

Denote  $\langle E_0, \dots, E_n \rangle \subset \mathbf{D}$  the full triangulated subcategory generated by the objects  $E_0, \dots, E_n$ . One proves [6, Lemma 1.58] that such a category is admissible.

Given a smooth algebraic variety  $X$  over a field  $k$ , denote  $\mathbf{D}^b(X)$  the bounded derived category of coherent sheaves. It is a  $k$ -linear triangulated category. Let  $\mathcal{E}$  be a vector bundle of rank  $r$  on  $X$ , and consider the associated projective bundle  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ . Denote  $\mathcal{O}_\pi(-1)$  the invertible line bundle on  $\mathbb{P}(\mathcal{E})$  of relative degree  $-1$ , such that  $\pi_* \mathcal{O}_\pi(1) = \mathcal{E}^*$ . One has [6, Corollary 8.36]:

**Theorem 2.2.** The category  $\mathbf{D}^b(\mathbb{P}(\mathcal{E}))$  has a semiorthogonal decomposition:

$$(7) \quad \mathbf{D}^b(\mathbb{P}(\mathcal{E})) = \langle \pi^* \mathbf{D}^b(X) \otimes \mathcal{O}_\pi(-r+1), \dots, \pi^* \mathbf{D}^b(X) \otimes \mathcal{O}_\pi(-1), \pi^* \mathbf{D}^b(X) \rangle.$$

In particular, the projection functor  $\pi_* : \mathbf{D}^b(\mathbb{P}(\mathcal{E})) \rightarrow \mathbf{D}^b(X)$  is surjective.

We also need some basic facts about generators in triangulated categories (see [10]).

**Definition 2.5.** Let  $\mathbf{D}$  be a  $k$ -linear triangulated category. An object  $C$  of  $\mathbf{D}$  is called *compact* if for any coproduct of objects one has  $\mathrm{Hom}_{\mathbf{D}}(C, \coprod_{\lambda \in \Lambda} X_\lambda) = \coprod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathbf{D}}(C, X_\lambda)$ .

**Definition 2.6.** A  $k$ -linear triangulated category  $\mathbf{D}$  is called *compactly generated* if  $\mathbf{D}$  contains small coproducts, and there exists a small set  $\mathbf{T}$  of compact objects of  $\mathbf{D}$ , such that  $\mathrm{Hom}_{\mathbf{D}}(\mathbf{T}, X) = 0$  implies  $X = 0$ . In other words, if  $X$  is an object of  $\mathbf{D}$ , and for every  $T \in \mathbf{T}$  one has  $\mathrm{Hom}_{\mathbf{D}}(T, X) = 0$ , then  $X$  must be the zero object.

**Definition 2.7.** Let  $\mathcal{D}$  be a compactly generated triangulated category. A set  $\mathcal{T}$  of compact objects of  $\mathcal{D}$  is called a generating set if  $\mathrm{Hom}_{\mathcal{D}}(\mathcal{T}, X) = 0$  implies  $X = 0$  and  $\mathcal{T}$  is closed under the shift functor, i.e.  $\mathcal{T} = \mathcal{T}[1]$ .

**Proposition 2.2.** [10, Example 1.10] Let  $X$  be a quasi-compact, separated scheme, and  $\mathcal{L}$  be an ample line bundle on  $X$ . Then the set  $\mathcal{L}^{\otimes m}, m \in \mathbb{Z}$  is a generating set for  $\mathcal{D}^b(X)$ .

### 3. Semiorthogonal decompositions for flag varieties

We observe first that the subcategory  $\langle \mathcal{L} \rangle$  of  $\mathcal{D}^b(\mathbf{G}/\mathbf{B})$  generated by a line bundle  $\mathcal{L}$  is admissible: indeed, it follows from Section 2.4 that  $\langle \mathcal{L} \rangle$  is such once the bundle  $\mathcal{L}$  is exceptional, i.e.  $\mathrm{Hom}_{\mathbf{G}/\mathbf{B}}^{\bullet}(\mathcal{L}, \mathcal{L}) = k$ . The latter condition is equivalent to  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = 0$  for  $i > 0$ .

**Proposition 3.1.** One has  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = 0$  for  $i > 0$ .

*Proof.* We recall two proofs of this key fact. First, the flag variety  $\mathbf{G}/\mathbf{B}$  over  $k$  has a lift to the ring of second Witt vectors  $W_2(k)$  as  $\mathbf{G}/\mathbf{B}$  itself is obtained by a reduction from the Chevalley group scheme  $\mathbf{G}/\mathbb{B}$  over  $\mathbb{Z}$ . By [3], the Kodaira vanishing theorem holds for  $\mathbf{G}/\mathbf{B}$  and  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = H^i(\mathbf{G}/\mathbf{B}, \omega_{\mathbf{G}/\mathbf{B}}^{-1} \otimes \omega_{\mathbf{G}/\mathbf{B}}) = 0$  for  $i > 0$ , the canonical sheaf  $\omega_{\mathbf{G}/\mathbf{B}}$  being anti-ample.

However, given a variety  $X$  over  $k$ , the proof in [3] of the Hodge degeneration and of the consequent Kodaira vanishing requires, apart a lifting of  $X$  to  $W_2(k)$ , that the characteristic of  $k$  be bigger than the dimension of  $X$ ; thus, the above argument gives that  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = 0$  for  $i > 0$  only if  $p > \dim(\mathbf{G}/\mathbf{B})$ . The following well-known argument is more specific to the setting of flag varieties, and does not impose restrictions on the characteristic. Recall that a morphism  $f : X \rightarrow Y$  between two projective varieties is said to have *rational singularities* if  $f_*\mathcal{O}_X = \mathcal{O}_Y$  (such morphisms are also called *trivial* in [8]). Given a Schubert variety  $Z$  of the flag variety  $\mathbf{G}/\mathbf{B}$ , Theorem 4 of [11] states that the Demazure resolution  $d_Z : \mathcal{Z} \rightarrow Z$  is a trivial morphism. In particular, considering the Demazure resolution  $\mathcal{X}$  associated to the longest element of the Weyl group, one sees that the morphism  $d : \mathcal{X} \rightarrow \mathbf{G}/\mathbf{B}$  is trivial, i.e.  $d_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathbf{G}/\mathbf{B}}$ . On the other hand, the variety  $\mathcal{X}$  is isomorphic to a successive tower of  $\mathbb{P}^1$ -bundles over a point (cf. the end of subsequent Lemma 3.1); hence,  $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$  for  $i > 0$ . Therefore,  $H^i(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}) = H^i(\mathbf{G}/\mathbf{B}, d_*\mathcal{O}_{\mathcal{X}}) = H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$  for  $i > 0$ . □

**Lemma 3.1.** Consider the semiorthogonal decomposition of  $\mathcal{D}^b(\mathbf{G}/\mathbf{B}) = \langle \langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle^{\perp}, \langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle \rangle$ . Then the subcategory  $\langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle^{\perp} \subset \mathcal{D}^b(\mathbf{G}/\mathbf{B})$  is generated, as an admissible triangulated subcategory of  $\mathcal{D}^b(\mathbf{G}/\mathbf{B})$ , by acyclic line bundles  $\mathcal{L}_{\chi}$  with the following property: there exists a simple coroot  $\alpha^{\vee} \in R^{\vee}$ , such that  $\langle \chi + \rho, \alpha^{\vee} \rangle = 0$ .

**Remark 3.1.** The generating set of the subcategory  $\langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle^{\perp}$  in Lemma 3.1 is not at all minimal.

*Proof.* One direction is ensured by Theorem 2.1, (a): a line bundle  $\mathcal{L}_\chi$  with  $\langle \chi + \rho, \alpha^\vee \rangle = 0$  for some simple  $\alpha^\vee \in R^\vee$  is acyclic, i.e. belongs to the subcategory  $\langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle^\perp$  (cf. Remark 2.1).

The following argument was suggested to us by Roman Bezrukavnikov.

Given a simple root  $\alpha$ , let  $\mathbf{P}_\alpha$  denote the minimal parabolic subgroup of  $\mathbf{G}$  associated to  $\alpha$ . Denote  $\pi_\alpha : \mathbf{G}/\mathbf{B} \rightarrow \mathbf{G}/\mathbf{P}_\alpha$  the projection. Observe first that  $\langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle^\perp$  contains the subcategory generated by  $\pi_\alpha^* \pi_{\alpha*} \mathcal{F} \otimes \mathcal{L}_{-\rho}$ , where  $\mathcal{F} \in D^b(\mathbf{G}/\mathbf{B})$ . Indeed,

$$(8) \quad \text{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\mathcal{O}_{\mathbf{G}/\mathbf{B}}, \pi_\alpha^* \pi_{\alpha*} \mathcal{F} \otimes \mathcal{L}_{-\rho}) = \mathbb{H}^*(\mathbf{G}/\mathbf{P}_\alpha, \pi_{\alpha*} \mathcal{F} \otimes \pi_{\alpha*} \mathcal{L}_{-\rho}) = 0,$$

as  $\pi_{\alpha*} \mathcal{L}_{-\rho} = 0$ . Let  $\mathcal{C} \subset D^b(\mathbf{G}/\mathbf{B})$  be the thick triangulated category generated by  $\pi_\alpha^* \pi_{\alpha*} \mathcal{F} \otimes \mathcal{L}_{-\rho}$ , where  $\mathcal{F} \in D^b(\mathbf{G}/\mathbf{P}_\alpha)$  and  $\alpha$  runs over the set of all the simple roots. Consider its left orthogonal  ${}^\perp \mathcal{C} \subset D^b(\mathbf{G}/\mathbf{B})$ . Observe next that  $\mathcal{C}$  coincides with the triangulated subcategory generated by line bundles satisfying the condition of Lemma 3.1: indeed, given a simple root  $\alpha$ , any line bundle  $\mathcal{L}_\chi$  on  $\mathbf{G}/\mathbf{P}_\alpha$  satisfies  $\langle \chi, \alpha^\vee \rangle = 0$ , and by Theorem 2.2, the projection functor  $\pi_{\alpha*} : D^b(\mathbf{G}/\mathbf{B}) \rightarrow D^b(\mathbf{G}/\mathbf{P}_\alpha)$  is surjective. On the other hand, choosing an ample line bundle  $\mathcal{L}$  on  $\mathbf{G}/\mathbf{P}_\alpha$ , the category  $D^b(\mathbf{G}/\mathbf{P}_\alpha)$  is generated by the set  $\mathcal{L}^{\otimes m}, m \in \mathbb{Z}$  in virtue of Proposition 2.2. Hence,  $\mathcal{C} \subset \langle \mathcal{L}_\chi \rangle$  with  $\langle \chi + \rho, \alpha^\vee \rangle = 0$  for a simple coroot  $\alpha^\vee$ .

Thus, it will be sufficient to show that the category  ${}^\perp \mathcal{C}$ , which is the left orthogonal to  $\mathcal{C}$ , is generated by the structure sheaf  $\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ . To this end, observe that any object  $\mathcal{G}$  of  ${}^\perp \mathcal{C}$  belongs to  $\pi_\alpha^* D^b(\mathbf{G}/\mathbf{P}_\alpha)$  for each simple root  $\alpha$ . Indeed, by Serre duality

$$(9) \quad \text{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\mathcal{G}, \pi_\alpha^* \pi_{\alpha*} \mathcal{F} \otimes \mathcal{L}_{-\rho}) = \text{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\pi_{\alpha*} \mathcal{F}, \pi_{\alpha*}(\mathcal{G} \otimes \mathcal{L}_{-\rho})[\dim(\mathbf{G}/\mathbf{B})]^*) = 0;$$

since  $\mathcal{F}$  is arbitrary and functor  $\pi_{\alpha*}$  is surjective, it follows that  $\pi_{\alpha*}(\mathcal{G} \otimes \mathcal{L}_{-\rho}) = 0$ . By Theorem 2.2, this implies  $\mathcal{G} \otimes \mathcal{L}_{-\rho} \in \pi_\alpha^* D^b(\mathbf{G}/\mathbf{P}_\alpha) \otimes \mathcal{O}_{\pi_\alpha}(-1)$ , which is equivalent to  $\mathcal{G} \in \pi_\alpha^* D^b(\mathbf{G}/\mathbf{P}_\alpha)$ , the pullback functor  $\pi_\alpha^*$  commuting with tensor products. Indeed, for any simple root  $\alpha$ , the restriction of  $\mathcal{L}_{-\rho}$  to a fibre of  $\pi_\alpha$  has degree equal to  $-1$ , hence  $\mathcal{L}_\rho \otimes \mathcal{O}_{\pi_\alpha}(-1) \in \pi_\alpha^* D^b(\mathbf{G}/\mathbf{P}_\alpha)$ . In particular, for all  $\alpha$  the restriction of  $\mathcal{G}$  to each fibre of  $\pi_\alpha$  belongs to the subcategory generated by  $\langle \mathcal{O}_{\pi_\alpha} \rangle$ .

Finally, considering the Demazure resolution  $d : \mathcal{X} \rightarrow \mathbf{G}/\mathbf{B}$  associated to the longest element of the Weyl group of  $\mathbf{R}$ , one sees that the object  $d^* \mathcal{G} \in D^b(\mathcal{X})$  belongs to the subcategory  $\langle \mathcal{O}_{\mathcal{X}} \rangle$ : indeed,  $\mathcal{X}$  is isomorphic to a tower of successive  $\mathbb{P}^1$ -bundles over a point, the fibres of each such  $\mathbb{P}^1$ -fibration being isomorphic to a fibre of some  $\pi_\alpha$ . By the above, the restriction of  $d^* \mathcal{G}$  to each fibre of  $\pi_\alpha$  belongs to the subcategory generated by  $\langle \mathcal{O}_{\pi_\alpha} \rangle$ . Applying successively Theorem 2.2, one sees that  $d^* \mathcal{G}$  belongs to  $\langle \mathcal{O}_{\mathcal{X}} \rangle$ . Now, by Proposition 3.1 the Demazure resolution  $d : \mathcal{X} \rightarrow \mathbf{G}/\mathbf{B}$  is a trivial morphism. By the projection formula, one obtains  $d_* d^* \mathcal{G} = \mathcal{G} \otimes d_* \mathcal{O}_{\mathcal{X}} = \mathcal{G} \in \langle d_* \mathcal{O}_{\mathcal{X}} \rangle = \langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle$ .  $\square$

**Corollary 3.1.** *Consider the semiorthogonal decomposition of  $D^b(\mathbf{G}/\mathbf{B}) = \langle \langle \mathcal{L}_{-\rho} \rangle, {}^\perp \langle \mathcal{L}_{-\rho} \rangle \rangle$ . Then the category  ${}^\perp \langle \mathcal{L}_{-\rho} \rangle$  is generated, as an admissible triangulated subcategory of  $D^b(\mathbf{G}/\mathbf{B})$ , by the set of line bundles  $\mathcal{L}_\chi$ , where  $\langle \chi, \alpha^\vee \rangle = 0$  for some simple coroot  $\alpha^\vee \in R^\vee$ .*

*Proof.* Let  $\mathcal{E} \in {}^\perp \langle \mathcal{L}_{-\rho} \rangle$ , then by Serre duality

$$(10) \quad \begin{aligned} \mathrm{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\mathcal{E}, \mathcal{L}_{-\rho}) &= \mathrm{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\mathcal{L}_{-\rho}, \mathcal{E} \otimes \mathcal{L}_{-2\rho}[\dim(\mathbf{G}/\mathbf{B})])^* = \\ &\mathrm{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\mathcal{O}_{\mathbf{G}/\mathbf{B}}, \mathcal{E} \otimes \mathcal{L}_{-\rho}[\dim(\mathbf{G}/\mathbf{B})])^* = 0. \end{aligned}$$

Therefore, up to a twist by the line bundle  $\mathcal{L}_\rho$ , the category  ${}^\perp \langle \mathcal{L}_{-\rho} \rangle$  is equivalent to the subcategory  $\langle \mathcal{O}_{\mathbf{G}/\mathbf{B}} \rangle^\perp$ . Lemma 3.1 implies the statement.  $\square$

#### 4. The Steinberg line bundle

Consider the admissible subcategory  $\langle \mathcal{L}_{-\rho} \rangle$  of  $D^b(\mathbf{G}/\mathbf{B})$ . It follows from the above that the isomorphism  $F_*^n \mathcal{L}_{-\rho} = \mathrm{St}_q \otimes \mathcal{L}_{-\rho}$  is equivalent to the following statement:

**Theorem 4.1.** *One has  $F_*^n \mathcal{L}_{-\rho} \subset \langle \mathcal{L}_{-\rho} \rangle$ .*

*Proof.* By Corollary 3.1, the fact that the bundle  $F_*^n \mathcal{L}_{-\rho}$  belongs to the subcategory  $\langle \mathcal{L}_{-\rho} \rangle \subset D^b(\mathbf{G}/\mathbf{B})$  is equivalent to saying that  $F_*^n \mathcal{L}_{-\rho}$  is right orthogonal to  $\langle \mathcal{L}_\chi \rangle$ , where  $\langle \chi, \alpha^\vee \rangle = 0$  for some simple coroot  $\alpha^\vee \in R^\vee$ . In other words, one has to ensure that

$$(11) \quad \mathrm{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\mathcal{L}_\chi, F_*^n \mathcal{L}_{-\rho}) = \mathbb{H}^*(\mathbf{G}/\mathbf{B}, \mathcal{L}_{-p^n \chi - \rho}) = 0.$$

By Theorem 2.1 (a), the line bundle  $\mathcal{L}_\mu$  is acyclic if  $\langle \mu + \rho, \alpha^\vee \rangle = 0$  for some simple coroot  $\alpha^\vee$ . Taking  $\mu = -p^n \chi - \rho$ , one obtains  $\langle \mu + \rho, \alpha^\vee \rangle = \langle -p^n \chi - \rho + \rho, \alpha^\vee \rangle = -p^n \langle \chi, \alpha^\vee \rangle = 0$ . Hence, the bundle  $\mathcal{L}_{-p^n \chi - \rho}$  is acyclic, and (11) holds.  $\square$

**Corollary 4.1.** *Let  $\mathcal{L}_\lambda$  be a line bundle on  $\mathbf{G}/\mathbf{B}$  and assume given an isomorphism  $F_*^n \mathcal{L}_\lambda = \mathbf{V}_\lambda \otimes \mathcal{L}_\lambda$  for some vector space  $\mathbf{V}_\lambda$ . Then  $\lambda = -\rho$ , and  $\mathbf{V}_\lambda = \mathrm{St}_q$ .*

*Proof.* Tensoring semiorthogonal decomposition  $D^b(\mathbf{G}/\mathbf{B}) = \langle \langle \mathcal{L}_{-\rho} \rangle, {}^\perp \langle \mathcal{L}_{-\rho} \rangle \rangle$  with  $\mathcal{L}_{\lambda+\rho}$ , one obtains  $D^b(\mathbf{G}/\mathbf{B}) = \langle \langle \mathcal{L}_\lambda \rangle, {}^\perp \langle \mathcal{L}_{-\rho} \rangle \otimes \mathcal{L}_{\lambda+\rho} \rangle$ . By Corollary 3.1, the admissible subcategory  $\langle {}^\perp \langle \mathcal{L}_{-\rho} \rangle \otimes \mathcal{L}_{\lambda+\rho} \rangle$  is generated by the set of line bundles  $\mathcal{L}_\mu$ , such that  $\langle \mu - \lambda - \rho, \alpha^\vee \rangle = 0$  for some simple coroot  $\alpha^\vee$ . By Theorem 4.1, the condition  $F_*^n \mathcal{L}_\lambda = \mathbf{V}_\lambda \otimes \mathcal{L}_\lambda$  is equivalent to  $F_*^n \mathcal{L}_\lambda \subset \langle \mathcal{L}_\lambda \rangle$ , thus

$$(12) \quad \mathrm{Hom}_{\mathbf{G}/\mathbf{B}}^\bullet(\mathcal{L}_\mu, F_*^n \mathcal{L}_\lambda) = \mathbb{H}^*(\mathbf{G}/\mathbf{B}, \mathcal{L}_{\lambda - p^n \mu - p^n \lambda - p^n \rho}) = 0,$$

should hold for all weights  $\mu$ , such that  $\langle \mu - \lambda - \rho, \alpha^\vee \rangle = 0$  for some simple coroot  $\alpha^\vee$ . On the other hand, by Proposition 2.1, the second equality in (12) implies



$$(13) \quad \langle \lambda - p^n \mu - p^n \lambda - p^n \rho + \rho, \beta^\vee \rangle = 0,$$

for some coroot  $\beta^\vee$ . Equivalently, the above condition reads

$$(14) \quad p^n \langle \mu, \beta^\vee \rangle = (1 - p^n) \cdot \langle \lambda + \rho, \beta^\vee \rangle.$$

Assume that  $\lambda + \rho \neq 0$ . Since the weight  $\lambda + \rho$  belongs to the subcategory  ${}^\perp \langle \mathcal{L}_{-\rho} \rangle \otimes \mathcal{L}_{\lambda+\rho}$ , equality (14) should hold for  $\mu = \lambda + \rho$ . Assume first that  $\langle \lambda + \rho, \beta^\vee \rangle \neq 0$ . Putting  $\mu = \lambda + \rho$  in (14), one obtains  $(2p^n - 1) \cdot \langle \lambda + \rho, \beta^\vee \rangle = 0$ , which is a contradiction.

In case  $\langle \lambda + \rho, \beta^\vee \rangle = 0$ , from (14) one obtains  $\langle \mu, \beta^\vee \rangle = 0$ , and hence  $\langle \mu - \lambda - \rho, \beta^\vee \rangle = 0$ . Recall that the latter equality should hold for all the weights  $\mu$ , such that  $\langle \mu - \lambda - \rho, \alpha^\vee \rangle = 0$  for some simple coroot  $\alpha^\vee$ . Writing down  $\beta^\vee$  as a sum of simple coroots, one sees that one can find a weight  $\mu$ , such that  $\mu - \lambda - \rho$  is orthogonal to all but one of the simple coroots in the decomposition of  $\beta^\vee$ . Hence,  $\langle \mu - \lambda - \rho, \beta^\vee \rangle \neq 0$ , which is again a contradiction.  $\square$

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MATH. INSTITUT, HEINRICH-HEINE-UNIVERSITÄT, D-40204 DÜSSELDORF, GERMANY

and

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, MOSCOW, RUSSIA

E-mail address: alexander.samokhin@gmail.com